

Quantum conductance problems and the Jacobi ensemble

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2006 J. Phys. A: Math. Gen. 39 6861

(<http://iopscience.iop.org/0305-4470/39/22/004>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.105

The article was downloaded on 03/06/2010 at 04:35

Please note that [terms and conditions apply](#).

Quantum conductance problems and the Jacobi ensemble

P J Forrester

Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia

Received 10 January 2006

Published 16 May 2006

Online at stacks.iop.org/JPhysA/39/6861

Abstract

In one-dimensional transport problems, the scattering matrix S is decomposed into a block structure corresponding to reflection and transmission matrices at the two ends. For S a random unitary matrix, the singular value probability distribution function of these blocks is calculated. The same is done when S is constrained to be symmetric, or to be self-dual quaternion real, as is relevant in the presence of particular time reversal symmetries. The latter results are shown to be very similar to those obtained under the (unphysical) assumption that S has real elements, or has real quaternion elements, respectively. Three methods are used: metric forms, a variant of the Ingham–Seigel matrix integral and a theorem specifying the Jacobi random matrix ensemble in terms of Wishart distributed matrices.

PACS numbers: 73.23.–b, 05.60.Gg

1. Introduction

In mesoscopic physics, the conductance of certain quasi one-dimensional wires containing scattering impurities exhibits the phenomenon known as universal conductance fluctuations. As first predicted theoretically [1, 11], and soon after observed experimentally [18], for such wires the variance of the conductance is of the order $(e^2/h)^2$, independent of sample size or disorder strength. Furthermore, the variance decreases by precisely a factor of 2 if time reversal symmetry is broken by a magnetic field. This conductance problem is fundamental for its relation to the Landauer scattering theory of electronic conduction (see [4] and references therein) and to time reversal symmetry. It is further fundamental for its relation to random matrix theory. Let us revise these points by way of background and motivation for the specific problem of this paper.

In the theoretical description of the conductance problem, basic quantities are the electron fluxes at the left- and right-hand edges of the wire. These are specified by an n -component vector \vec{I} and an m -component vector \vec{I}' specifying the complex amplitudes of the available plane wave states travelling into the left and right sides of the wire, respectively, as well as an

n -component vector \vec{O} and an m -component vector \vec{O}' for the same states travelling out of the left and right sides of the wire. For definiteness, it will be assumed that $n \geq m$.

By definition, the scattering matrix S relates the flux travelling into the conductor to that travelling out,

$$S \begin{bmatrix} \vec{I} \\ \vec{I}' \end{bmatrix} := \begin{bmatrix} \vec{O} \\ \vec{O}' \end{bmatrix}. \quad (1.1)$$

The scattering matrix is further decomposed in terms of reflection and transmission matrices by

$$S = \begin{bmatrix} r_{n \times n} & t'_{n \times m} \\ t_{m \times n} & r'_{m \times m} \end{bmatrix}. \quad (1.2)$$

Flux conservation requires

$$|\vec{I}|^2 + |\vec{I}'|^2 = |\vec{O}|^2 + |\vec{O}'|^2,$$

and this implies that S must be unitary. Furthermore, by relating S to a Hamiltonian [17, 19], one can argue that S must be symmetric if the system has a time reversal symmetry with $T^2 = 1$ and a self-dual quaternion matrix when there is a time reversal symmetry with $T^2 = -1$.

The immediate relevance of the above formalism is seen by invoking the Landauer scattering theory of electronic conduction. According to this formalism, the conductance G is given in terms of the transmission matrix $t_{m \times n}$ (or $t'_{n \times m}$) by the so-called two-probe Landauer formula

$$G/G_0 = \text{Tr}(t^\dagger t) = \text{Tr}(t'^\dagger t'), \quad (1.3)$$

where $G_0 = 2e^2/h$ is twice the fundamental quantum unit of conductance. Thus, to compute G it suffices to know the distribution of the eigenvalues of $t^\dagger t$ (or $t'^\dagger t'$). In fact, the matrix S can be decomposed in a form which isolates these eigenvalues.

For definiteness, suppose there is no time reversal symmetry, so S is a general unitary matrix. Each block of S can then be decomposed according to a general singular value decomposition. For example, for the block $r_{n \times n}$ we have

$$r_{n \times n} = U_r \Lambda_r V_r^\dagger,$$

where U_r, V_r are unitary matrices and Λ_r is a rectangular diagonal matrix with entries equal to the positive square roots of the eigenvalues of $r^\dagger r$ (these eigenvalues are between 0 and 1 since $r^\dagger r + t^\dagger t = I_{n \times n}$). The unitarity of S inter-relates the matrices $U_r, U_{r'}, \dots$ (see e.g. [16]) and implies the decomposition

$$S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} V_r^\dagger & 0 \\ 0 & V_{r'}^\dagger \end{bmatrix}, \quad (1.4)$$

where

$$L := \begin{bmatrix} \sqrt{1 - \Lambda_t \Lambda_t^T} & i\Lambda_t \\ i\Lambda_t^T & \sqrt{1 - \Lambda_t^T \Lambda_t} \end{bmatrix}. \quad (1.5)$$

Symmetries relate $V_r, V_{r'}$ to $U_r, U_{r'}$. As already remarked, in the case of a time reversal symmetry $T^2 = 1$, S must be symmetric, while for $T^2 = -1$, S must be self-dual quaternion real. These symmetries require that

$$V_r^\dagger = U_r^T, \quad V_{r'}^\dagger = U_{r'}^T \quad \text{and} \quad V_r^\dagger = U_r^D, \quad V_{r'}^\dagger = U_{r'}^D, \quad (1.6)$$

respectively. In (1.6), the operation D , for an $n \times n$ matrix A with real quaternion elements regarded as a $2n \times 2n$ matrix with complex elements, is specified by

$$A^D = Z_{2n} A^T Z_{2n}^{-1}, \quad Z_{2n} := I_n \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Random impurities in the quantum wire effectively mean that the scattering matrix itself is random. There is a natural measure on the space of unitary matrices, the Haar measure, in which all matrices occur with equal probability. Adopting this measure, the non-zero elements of Λ_t , which in turn are equal to the square root of the non-zero eigenvalues of $t^\dagger t$ and are thus in the interval $(0, 1)$, have a joint probability density function (p.d.f.) proportional to

$$\prod_{j=1}^m \lambda_j^{\beta\alpha} \prod_{1 \leq j < k \leq m} |\lambda_k^2 - \lambda_j^2|^\beta, \quad \alpha := n - m + 1 - 1/\beta, \quad (1.7)$$

where $\beta = 1, 2, 4$ according to S being constrained to be symmetric, no constraints, or constrained to be self-dual quaternion. In its full generality, this result is stated without proof in [4, equation (2.9)], where it is attributed to Brouwer. In the case $n = m$, it was derived in [2, 14]. With knowledge of (1.7), it can be deduced from (1.3) that with α fixed [3]

$$\lim_{m, n \rightarrow \infty} \text{Var}(G/G_0) = \frac{1}{8\beta}, \quad (1.8)$$

which is in quantitative agreement with the experiment.

However, the issue of fluctuation formulae is not our concern here. Rather, we seek to specify three distinct derivations of (1.7). These are the method of metric forms, an application of a version of the Ingham–Seigel integral (following [7]) and use of a theorem from mathematical statistics on the distribution of matrices of the form

$$J := (C + D)^{-1/2} C (C + D)^{-1/2} \quad (1.9)$$

with C and D further specified in terms of Wishart matrices (following [5]). In fact, only the first of these applies to the three symmetry classes $\beta = 1, 2, 4$, while the other two require that $\beta = 2$. However, these latter two deviations allow for an easy extension of (1.7) in the case $\beta = 2$ to a more general decomposition of S than (1.2), which has relevance to a three-lead quantum dot problem. They also give analogous results in the case of unitary matrices with real elements, or with real quaternion elements. These turn out to be closely related to (1.7) with $\beta = 1$ and 4, respectively, notwithstanding that the classes of unitary matrices under consideration are not physical when viewed as scattering matrices. We remark that aspects of statistical properties of the block decomposition of real orthogonal matrices (i.e. unitary matrices with real elements) have been previously given in [20].

A noteworthy feature of the resulting eigenvalue p.d.f.s is that they can all be written in the form

$$\frac{1}{C} \prod_{l=1}^N x_l^a (1 - x_l)^b \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta, \quad (1.10)$$

where $0 < x_j < 1$ ($j = 1, \dots, N$) and C denotes the normalization (to relate (1.7) to this form change variables $\lambda_j^2 = x_j$). The p.d.f. is said to specify the Jacobi ensemble or random matrices (see e.g. [6]). The Jacobi ensemble with $\beta = 1$, under the name of the generalized beta distribution, is well known in mathematical statistics as the eigenvalue distribution of the matrix (1.9). The third of our derivations uses this result and thereby relates the seemingly two different problems in random matrix theory.

2. Metric forms

Let $X = [x_{jk}]_{j,k=1,\dots,N}$ be an $N \times N$ matrix. Let $\{x_\mu\}$, where μ labels both rows and columns, be the set of independent real and imaginary parts in X . The metric form of the line element ds is defined by

$$(ds)^2 := \text{Tr}(dX dX^\dagger) = \sum_{\mu \text{ diag.}} (dx_\mu)^2 + 2 \sum_{\substack{\mu \text{ upper} \\ \text{triangular}}} (dx_\mu)^2.$$

The corresponding volume measure is

$$(dX) = \bigwedge_{\mu \text{ diag.}} dx_\mu \bigwedge_{\substack{\mu \text{ upper} \\ \text{triangular}}} dx_\mu. \quad (2.1)$$

Suppose now a change of variables $\{x_\mu\} \mapsto \{y_\mu\}$ is made such that $(ds)^2$ is a symmetric quadratic form in $\{dy_\mu\}$,

$$(ds)^2 = \sum_{\mu, \nu} g_{\mu, \nu} dy_\mu dy_\nu, \quad g_{\mu, \nu} = g_{\nu, \mu}.$$

The corresponding volume measure is then (see e.g. [12])

$$(\det[g_{\mu, \nu}])^{1/2} \bigwedge_{\mu} dy_\mu. \quad (2.2)$$

In [6], this formalism has been used to derive the result (1.7) in the case $\beta = 2$. Here the details will be given for the case $\beta = 1$, when (1.4) reads

$$S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} U_r^T & 0 \\ 0 & U_{r'}^T \end{bmatrix}. \quad (2.3)$$

For a general matrix A , let $\delta A := A^\dagger dA$, where dA denotes the matrix of differentials of the elements of A . Using the fact that for A unitary $(\delta A)^\dagger = -\delta A$, it follows from (2.3) that

$$\begin{bmatrix} U_r^\dagger & 0 \\ 0 & U_{r'}^\dagger \end{bmatrix} dS \begin{bmatrix} \bar{U}_r & 0 \\ 0 & \bar{U}_{r'} \end{bmatrix} = \begin{bmatrix} \delta U_r & 0 \\ 0 & \delta U_{r'} \end{bmatrix} L + dL - L \begin{bmatrix} \delta \bar{U}_r & 0 \\ 0 & \delta \bar{U}_{r'} \end{bmatrix}. \quad (2.4)$$

In $\text{Tr}(dS dS^\dagger)$, the right-hand side of (2.4) can effectively be substituted for dS . Doing this and simplifying using

$$\text{Tr} \left(\begin{bmatrix} \delta U_r & 0 \\ 0 & \delta U_{r'} \end{bmatrix} L^\dagger dL \right) = 0$$

gives

$$\text{Tr}(dS dS^\dagger) = \text{Tr} A_1 \bar{A}_1 + \text{Tr}(A_2 + \bar{A}_2) + \text{Tr}(A_3 \bar{A}_3) + \text{Tr}(dL dL^\dagger), \quad (2.5)$$

where, with I denoting the identity matrix,

$$\begin{aligned} A_1 &= \sqrt{I - \Lambda_t \Lambda_t^T} \delta \bar{U}_r - \delta U_r \sqrt{I - \Lambda_t \Lambda_t^T}, \\ A_2 &= (\Lambda_t \delta \bar{U}_{r'} - \delta U_r \Lambda_t) (\Lambda_t^T \delta U_r - \delta \bar{U}_{r'} \Lambda_t^T), \\ A_3 &= \sqrt{I - \Lambda_t \Lambda_t^T} \delta \bar{U}_{r'} - \delta U_{r'} \sqrt{I - \Lambda_t \Lambda_t^T}. \end{aligned}$$

We recall that the diagonal elements of Λ_t are the positive square roots of the eigenvalues of $t^\dagger t$. Because $t = t_{m \times n}$ and thus has rank m , $t^\dagger t$ must have $n - m$ zero eigenvalues, so with the diagonal elements of $\Lambda_t \Lambda_t^T$ denoted by λ_j ($j = 1, \dots, n$), as is consistent with the

notation of (1.7), we have $\lambda_{m+1} = \dots = \lambda_n = 0$. Using this fact, we see that in component form the four terms in (2.5), T_1, \dots, T_4 say, can be expanded to read

$$\begin{aligned}
 T_1 &= \sum_{k=1}^n (1 - \lambda_k^2) |(\delta \bar{U}_r)_{kk} - (\delta U_r)_{kk}|^2 + \sum_{1 \leq k < l \leq n} \frac{1}{2} (\sqrt{1 - \lambda_l^2} + \sqrt{1 - \lambda_k^2})^2 |(\delta \bar{U}_r)_{lk} - (\delta U_r)_{lk}|^2 \\
 &\quad + \left(\sum_{1 \leq k < l \leq m} + \sum_{k=1}^m \sum_{l=m+1}^n \right) \frac{1}{2} (\sqrt{1 - \lambda_l^2} - \sqrt{1 - \lambda_k^2})^2 |(\delta \bar{U}_r)_{lk} + (\delta U_r)_{lk}|^2, \\
 T_2 &= 2 \left(\sum_{k=1}^m \lambda_k^2 |(\delta \bar{U}_{r'})_{kk} - (\delta U_{r'})_{kk}|^2 + \sum_{1 \leq k < l \leq m} \left\{ \frac{1}{2} (\lambda_l + \lambda_k)^2 |(\delta \bar{U}_{r'})_{lk} - (\delta U_{r'})_{lk}|^2 \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} (\lambda_l - \lambda_k)^2 |(\delta \bar{U}_{r'})_{lk} + (\delta U_{r'})_{lk}|^2 \right\} + \sum_{k=1}^m \sum_{l=m+1}^n \lambda_k^2 |(\delta U_{r'})_{lk}|^2 \right), \\
 T_3 &= \sum_{k=1}^m (1 - \lambda_k^2) |(\delta \bar{U}_{r'})_{kk} - (\delta U_{r'})_{kk}|^2 + \sum_{1 \leq k < l \leq m} \frac{1}{2} (\sqrt{1 - \lambda_l^2} + \sqrt{1 - \lambda_k^2})^2 \\
 &\quad \times |(\delta \bar{U}_{r'})_{lk} - (\delta U_{r'})_{lk}|^2 + \frac{1}{2} (\sqrt{1 - \lambda_l^2} - \sqrt{1 - \lambda_k^2})^2 |(\delta \bar{U}_{r'})_{lk} + (\delta U_{r'})_{lk}|^2, \\
 T_4 &= 2 \sum_{j=1}^m \frac{(d\lambda_j)^2}{1 - \lambda_j^2}.
 \end{aligned}$$

In general, a symmetric unitary matrix has the same number of independent elements as a real symmetric matrix of the same rank, so S has $\frac{1}{2}(m+n)(m+n+1)$ independent elements. We thus seek this same number of independent differentials in T_1, \dots, T_4 . These are

$$(\delta U_r)_{kk}^{(i)}, \quad 1 \leq k \leq n, \quad (\delta U_{r'})_{kk}^{(i)}, \quad 1 \leq k \leq m,$$

and

$$(\delta U_r)_{lk}^{(r)}, \quad (\delta U_r)_{lk}^{(i)}, \quad (\delta U_r)_{lk}^{(r)}, \quad (\delta U_r)_{lk}^{(i)}, \quad 1 \leq k < l \leq m,$$

as well as

$$(\delta U_r)_{lk}^{(r)}, \quad (\delta U_r)_{lk}^{(i)}, \quad 1 \leq k \leq m \quad \text{and} \quad m+1 \leq l \leq n,$$

and

$$(\delta U_r)_{lk}^{(i)}, \quad m+1 \leq l < k \leq n, \quad d\lambda_j, \quad 1 \leq j \leq m,$$

which indeed tally to $\frac{1}{2}(m+n)(m+n+1)$.

We see from the expressions for T_1, T_2, T_3 that the contribution to the metric form from the differentials with subscripts lk such that $1 \leq k < l \leq m$ is

$$\begin{aligned}
 4 \sum_{1 \leq k < l \leq m} & (1 + \sqrt{1 - \lambda_l^2} \sqrt{1 - \lambda_k^2}) ((\delta U_r^{(i)})_{lk}^2 + (\delta U_{r'}^{(i)})_{lk}^2) \\
 & + (1 - \sqrt{1 - \lambda_l^2} \sqrt{1 - \lambda_k^2}) ((\delta U_r^{(r)})_{lk}^2 + (\delta U_{r'}^{(r)})_{lk}^2) \\
 & - 2\lambda_l \lambda_k (\delta U_{r'}^{(r)})_{lk} (\delta U_r^{(r)})_{lk} + 2\lambda_l \lambda_k (\delta U_{r'}^{(i)})_{lk} (\delta U_r^{(i)})_{lk}.
 \end{aligned} \tag{2.6}$$

Setting $\sqrt{1 - \lambda_l^2} \sqrt{1 - \lambda_k^2} =: a$ for notational convenience, this portion of the metric form contributes to $(\det[g_{jk}])^{1/2}$ in (2.2) 2×2 block factors which is proportional to

$$\prod_{k < l} \begin{vmatrix} 1+a & -\lambda_l \lambda_k \\ -\lambda_l \lambda_k & 1+a \end{vmatrix}^{1/2} \begin{vmatrix} 1-a & \lambda_l \lambda_k \\ \lambda_l \lambda_k & 1-a \end{vmatrix}^{1/2} = \prod_{k < l} |\lambda_l^2 - \lambda_k^2|. \tag{2.7}$$

For $m + 1 \leq k < l \leq n$, the coefficient of $(\delta U_r)_{lk}^{(i)}$ in T_1 is independent of λ s and so for the present purposes can be ignored. For $1 \leq k \leq m, m + 1 \leq l \leq n$, we see from T_1 and T_2 that the corresponding contribution to the metric form is

$$4 \sum_{k=1}^m \sum_{l=m+1}^n (1 + \sqrt{1 - \lambda_k^2}) ((\delta U_r)_{lk}^{(i)})^2 + (1 - \sqrt{1 - \lambda_k^2}) ((\delta U_r)_{lk}^{(v)})^2.$$

This contributes to the volume form a factor proportional to

$$\prod_{k=1}^m \prod_{l=m+1}^n \lambda_k = \prod_{k=1}^m \lambda_k^{n-m}. \tag{2.8}$$

We read off that the contribution to $(\det[g_{jk}])^{1/2}$ from the coefficients of the terms $(d\lambda_j)^2$ in T_4 is proportional to

$$\prod_{k=1}^m \frac{1}{(1 - \lambda_k^2)^{1/2}}. \tag{2.9}$$

It remains to calculate the contribution from the differentials on the diagonal. For $m + 1 \leq k \leq n$, the coefficient of $(\delta U_r)_{kk}^{(i)}$ is a constant so these differentials can be ignored. We read off from T_1, T_2, T_3 that the contribution to the metric form from the remaining differentials is

$$2 \sum_{k=1}^m (2 - \lambda_k^2) ((\delta U_r)_{kk}^{(i)})^2 + (\delta U_{r'}_{kk}^{(i)})^2 + 2\lambda_k^2 (\delta U_r)_{kk}^{(i)} (\delta U_{r'}_{kk}^{(i)}).$$

The contribution to the volume form is thus proportional to

$$\prod_{k=1}^m \left| \begin{matrix} 2 - \lambda_k^2 & \lambda_k^2 \\ \lambda_k^2 & 2 - \lambda_k^2 \end{matrix} \right|^{1/2} \propto \prod_{k=1}^m (1 - \lambda_k^2)^{1/2}. \tag{2.10}$$

Multiplying together (2.7)–(2.10) gives (1.7) in the case $\beta = 1$.

3. Matrix integrals

Let

$$I_{m,n}^{(2)}(Q_m) := \int e^{\frac{i}{2} \text{Tr}(H_m Q_m)} (\det(H_m - \mu I_m))^{-n} (dH_m), \tag{3.1}$$

where H_m, Q_m are $m \times m$ Hermitian matrices, and suppose $n \geq m, \text{Im } \mu > 0$. For Q_m positive definite, it has been proved by Fyodorov [7] that

$$I_{m,n}^{(2)}(Q_m) = \frac{2^m \pi^{m(m+1)/2} i^m (-1)^{m(m-1)/2}}{\prod_{j=0}^{m-1} \Gamma(n - j)} \left(\det \left(\frac{i}{2} Q_m \right) \right)^{n-m} e^{\frac{i}{2} \mu \text{Tr } Q_m}. \tag{3.2}$$

(Here we have taken $(dH_m) := \prod_{j=1}^m dh_{jj}^{(i)} \prod_{j < k} dh_{jk}^{(v)} dh_{jk}^{(i)}$ which differs by a factor of 2 in the product over $j < k$ to the convention adopted in [7].) This matrix integral may be regarded as being of the type first evaluated by Ingham and Siegel (see [7] and references therein). In a subsequent work [8], (3.1) was used to derive (1.7) in the case $\beta = 2$. Here we will show that this derivation can be used to derive a generalization of (1.7) in the case $\beta = 2$, and this generalization can be further extended to the cases $\beta = 1$ and $\beta = 4$, using suitable variants of (3.1), where now $\beta = 1$ and 4 refers to the decomposition (1.2) with S having real and real quaternion elements, respectively.

Define by $I_{m,n}^{(1)}(Q_m)$ the matrix integral (3.1) with H_m and Q_m now real symmetric. Also, define by $I_{m,n}^{(4)}(Q_m)$ the same matrix integral but with H_m and Q_m now self-dual quaternion Hermitian matrices (such matrices regarded as $2m \times 2m$ Hermitian matrices are doubly degenerate; adopt the convention that the operations Tr and \det include only distinct eigenvalues). In [7], it is remarked that the method of derivation given therein to deduce the evaluation of $I_{m,n}^{(2)}(Q_m)$ can also be used to deduce the evaluation of $I_{m,n}^{(1)}(Q_m)$, which reads

$$I_{m,n/2}^{(1)}(Q_m) = \frac{2^m \pi^{m(m+3)/2} i^{m(m+1)/2}}{\prod_{j=0}^{m-1} \Gamma((n-j)/2)} \left(\det \left(\frac{i}{2} Q_m \right) \right)^{(n-m-1)/2} e^{\frac{i}{2} \mu \text{Tr} Q_m}. \tag{3.3}$$

Applying the same method to $I_{m,n}^{(4)}(Q_m)$ gives

$$I_{m,2n}^{(4)}(Q_m) = \frac{(2i)^m \pi^{m^2}}{\prod_{j=0}^{m-1} \Gamma(2(n-j))} \left(\det \left(\frac{i}{2} Q_m \right) \right)^{2(n-m+1/2)} e^{\frac{i}{2} \mu \text{Tr} Q_m}. \tag{3.4}$$

Hence, for all three cases we have

$$I_{m,\beta n/2}^{(\beta)}(Q_m) = C_{m,n}^{(\beta)} (\det(Q_m))^{(\beta/2)(n-m+1-2/\beta)} e^{\frac{i}{2} \mu \text{Tr} Q_m}, \tag{3.5}$$

where $C_{m,n}^{(\beta)}$ is independent of Q_m .

Let U be an $N \times N$ random unitary matrix, with real ($\beta = 1$), complex ($\beta = 2$) and real quaternion ($\beta = 4$), chosen with Haar measure. Generalizing (1.2), decompose U into blocks

$$U = \begin{bmatrix} A_{n_1 \times n_2} & C_{n_1 \times (N-n_2)} \\ B_{(N-n_1) \times n_2} & D_{(N-n_1) \times (N-n_2)} \end{bmatrix}. \tag{3.6}$$

Since U is unitary, we require

$$AA^\dagger + CC^\dagger = I_{n_1}, \tag{3.7}$$

together with three similar equations involving B and D which given A and C can always be satisfied. The idea of [20] is to regard (3.7) as a constraint in the space of general rectangular matrices A, C with entries of the type required by the index β . Thus, in this viewpoint the distribution of A is given by

$$\int \delta(AA^\dagger + CC^\dagger - I_{n_2}) (dC). \tag{3.8}$$

In (3.8), the delta function is a product of scalar delta functions, one for each independent real and imaginary component of $A^\dagger A + C^\dagger C - I_{n_2}$. It is proportional to the matrix integral

$$\int e^{-i \text{Tr}(H(AA^\dagger + CC^\dagger - I_{n_2}))} (dH), \tag{3.9}$$

where H is a Hermitian matrix with elements of the type β . Following [8], we would like to substitute (3.9) for the delta function in (3.8) and change the order of integration. The integrations over (dC) are simply Gaussian integrals. For the resulting function of H to be integrable around $H = 0$, the replacement $H \mapsto H - i\mu I_{n_1}$ in the exponent of (3.9) must be made. Doing this and computing the Gaussian integrals gives that (3.8) is proportional to

$$\lim_{\mu \rightarrow 0^+} \int (\det(H - i\mu I_{n_1}))^{-\beta(N-n_2)/2} e^{i \text{Tr}(H(I_{n_1} - AA^\dagger))} (dH).$$

Evaluating the matrix integral using (3.5) shows that the distribution of A is proportional to

$$(\det(I_{n_1} - AA^\dagger))^{(\beta/2)(N-n_1-n_2+1-2/\beta)}. \tag{3.10}$$

Note that for this to be normalizable we must have

$$N - n_1 - n_2 \geq 0. \tag{3.11}$$

Suppose that in addition to (3.11) we have $n_1 \geq n_2$. Then A has rank n_1 and so AA^\dagger has $n_1 - n_2$ zero eigenvalues and (3.10) can be written as

$$\left(\det(I_{n_2} - A^\dagger A)\right)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}. \quad (3.12)$$

Setting $Y = A^\dagger A$, we know (see e.g. [6]) that

$$dA \propto (\det Y)^{(\beta/2)(n_1-n_2+1-2/\beta)}(dY), \quad (3.13)$$

so we have from (3.12) that the distribution of Y is proportional to

$$(\det Y)^{(\beta/2)(n_1-n_2+1-2/\beta)} \left(\det(I_{n_2} - Y)\right)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}. \quad (3.14)$$

Denote the eigenvalues of Y by y_1, \dots, y_{n_2} . Using the fact that the eigenvalue-dependent portion of the Jacobian for a Hermitian matrix, with elements of the type β , when changing variables to its eigenvalues and eigenvectors is $\prod_{j < k} |y_k - y_j|^\beta$, we read off from (3.14) that the eigenvalue p.d.f. of Y is proportional to

$$\prod_{j=1}^{n_2} y_j^{(\beta/2)(n_1-n_2+1-2/\beta)} (1 - y_j)^{(\beta/2)(N-n_1-n_2+1-2/\beta)} \prod_{j < k} |y_k - y_j|^\beta. \quad (3.15)$$

In the case

$$n_1 = n_2 = m, \quad N = n + m, \quad (3.16)$$

the matrix A in (3.6) coincides with r in (1.2), and y_j in (3.15) are then the eigenvalues of $r^\dagger r$. The non-zero values of the singular values of the submatrix t in (1.2) are λ_j s in (1.7). The decomposition (1.4) tells us that $y_j = 1 - \lambda_j^2$. Indeed, making this change of variable in (3.15), and making the substitutions (3.16), reclaims (1.7) in the case $\beta = 2$.

For $\beta = 2$ and general $n_1 \geq n_2$, $N \geq n_1 + n_2$, a result equivalent to (3.15) is derived in a recent work of Simon and Moustakas [15]. Motivated by a quantum dot problem with three leads, they decomposed the $N \times N$ scattering matrix S into a 3×3 block structure

$$S = \begin{bmatrix} r_{11} & t_{12} & t_{13} \\ t_{21} & r_{22} & t_{23} \\ t_{31} & t_{32} & r_{33} \end{bmatrix}, \quad (3.17)$$

where each r_{ii} is $N_i \times N_i$ and t_{ij} is $N_i \times N_j$, with $N = N_1 + N_2 + N_3$. For S a random unitary matrix with Haar measure, it is shown that the eigenvalues of $t_{12}^\dagger t_{12}$ have p.d.f. given by (3.15) with $\beta = 2$, $n_1 = N$, $n_2 = N_2$. This is consistent with our result because the distribution of S is unchanged by interchanging rows and columns. The first and second block columns in (3.17) can be interchanged, effectively giving the decomposition (3.6) with $t_{12} = A_{N_1 \times N_2}$.

We remark that for $y_j \ll 1$ the term involving $(1 - y_j)$ in (3.15) can be ignored. The resulting p.d.f. is then the $x_j \ll 1$ limit of the singular values p.d.f. of the matrix product $X^\dagger X$, where X is an $n_1 \times n_2$ random Gaussian matrix with real ($\beta = 1$), complex ($\beta = 2$) or real quaternion elements ($\beta = 4$) (see e.g. [6]). This is consistent with a recent result of Jiang [10], who quantifies the degree to which the entries of orthogonal or unitary random matrices can be approximated by entries of Gaussian random matrices with real or complex entries, respectively.

4. Random projections

In a recent work, Collins [5] computed the distribution of certain products of random orthogonal projections. Here this approach, suitably modified, will be used to derive (3.15). Knowledge is required of the following result [6, 13].

Proposition 4.1. *Let c and d be $n_1 \times m$ and $n_2 \times m$ Gaussian random matrices, all elements identically and independently distributed (i.i.d.) where $n_1, n_2 \geq m$, indexed by the parameter $\beta = 1, 2$ or 4 . The parameter β specifies that the elements are real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$). Let $C = c^\dagger c$, $D = d^\dagger d$. The distribution of $J := (C + D)^{-1/2} C (C + D)^{-1/2}$ is proportional to*

$$(\det J)^{a\beta/2} (\det(I - J))^{b\beta/2} \tag{4.1}$$

with

$$a = n_1 - m + 1 - 2/\beta, \quad b = n_2 - m + 1 - 2/\beta.$$

To make use of proposition 4.1, let X be a $N \times m$, $N \geq m$ i.i.d. Gaussian random matrix indexed by β . Define Q_1 to be the top $n \times N$ submatrix of an $N \times N$ unitary matrix indexed by β . Because $Q_1 Q_1^\dagger = I_n$, we have that $Q_1 X$ is distributed as an $n \times m$ i.i.d. Gaussian random matrix indexed by β (see [9, theorem 2.3.1]). Let Q_2 be the bottom $(N - n) \times N$ submatrix of U so that

$$Q_1^\dagger Q_1 + Q_2^\dagger Q_2 = I_N. \tag{4.2}$$

Because $Q_2 Q_2^\dagger = I_{N-n}$, $Q_2 X$ is distributed as an $(N - n) \times m$ i.i.d. Gaussian matrix indexed by β .

Now set

$$Y = X^\dagger Q_1^\dagger Q_1 X, \quad Z = X^\dagger Q_2^\dagger Q_2 X. \tag{4.3}$$

Denoting by $W_p^{(\beta)}(n)$ the (Wishart) distribution of the matrix product $A^\dagger A$, where A is an $n \times p$, ($n \geq p$) i.i.d. Gaussian random matrix indexed by β , we see that the distribution of Y is $W_m^{(\beta)}(n)$ while the distribution of Z is $W_m^{(\beta)}(N - n)$. The result of proposition 4.1 tells us that the eigenvalue p.d.f. of

$$(Y + Z)^{-1/2} Y (Y + Z)^{-1/2} = (X^\dagger X)^{-1/2} Y (X^\dagger X)^{-1/2} \tag{4.4}$$

is given by (4.1) with $n_1 = n$, $n_2 = N - n$.

According to the singular value decomposition, we can write

$$X = U_1 \Lambda U_2 \tag{4.5}$$

for U_1 an $N \times N$ unitary matrix, U_2 an $m \times m$ unitary matrix and Λ an $N \times m$ diagonal matrix, with diagonal entries equal to the positive square root of the eigenvalues of $X^\dagger X$. It follows from (4.5) that

$$(X^\dagger X)^{1/2} = U_2 [\Lambda]_{m,m} U_2, \tag{4.6}$$

where $[\Lambda]_{m,m}$ refers to the top $m \times m$ sub-block of Λ , while (4.6) and (4.5) together imply

$$X (X^\dagger X)^{-1/2} = [U_1]_{N,m} U_2^\dagger. \tag{4.7}$$

Substituting for Y in (4.4) according to (4.3), then using (4.7) to substitute for $X (X^\dagger X)^{-1/2}$ tells us that the distribution of

$$U_2 [U_1]_{N,m}^\dagger Q_1^\dagger Q_1 [U_1]_{N,m} U_2^\dagger \tag{4.8}$$

is given by (4.1). We note

$$U_1^\dagger [U_1]_{N,m} = \text{diag}(1, \dots, 1, 0, \dots, 0),$$

where there are m 1s. Since the distribution of Q_1 is unchanged upon multiplication by a unitary matrix, the distribution of (4.8) is the same as the distribution of

$$U_2 [Q_1^\dagger Q_1]_{m,m} U_2^\dagger. \tag{4.9}$$

In terms of the decomposition (3.6), this in turn has the same distribution as

$$A^\dagger A \Big|_{\substack{n_1 \mapsto n \\ n_2 \mapsto m}}. \quad (4.10)$$

Hence, we have the distribution of (4.10) given by (4.1) with

$$a = n - m + 1 - 2/\beta, \quad b = N - n - m + 1 - 2/\beta,$$

which is in precise agreement with (3.14) upon the replacements $n_1 \mapsto n$, $n_2 \mapsto m$ in the latter.

Acknowledgments

This work was supported by the Australian Research Council. Remarks of S Simon relating to the first draft of this work are appreciated.

References

- [1] Altshuler B L 1985 Fluctuations in the extrinsic conductivity of disordered conductors *JETP Lett.* **41** 648–51
- [2] Baranger H U and Mello P A 1994 Mesoscopic transport through chaotic cavities: a random S -matrix theory approach *Phys. Rev. Lett.* **73** 142–5
- [3] Beenakker C W J 1993 Universality in the random-matrix theory of quantum transport *Phys. Rev. Lett.* **70** 1155–8
- [4] Beenakker C W J 1997 Random-matrix theory of quantum transport *Rev. Mod. Phys.* **69** 731–808
- [5] Collins B 2005 Product of random projections, Jacobi ensembles and universality problems arising from free probability *Probab. Theory Relat. Fields* **133** 315–44
- [6] Forrester P J 2005 Log-gases and random matrices, www.ms.unimelb.edu.au/~matpjf/matpjf.html
- [7] Fyodorov Y V 2002 Negative moments of characteristic polynomials of random matrices: Ingham–Seigel integral as an alternative to Hubbard–Stratonovich transformation *Nucl. Phys. B* **621** 643–74
- [8] Fyodorov Y V and Sommers H-J 2003 Random matrices close to Hermitian or unitary: overview of methods and results *J. Phys. A: Math. Gen.* **36** 3303–47
- [9] Gupta A K and Nagar D K 1999 *Matrix Variate Distributions* (Boca Raton, FL: Chapman and Hall/CRC Press)
- [10] Jiang T F 2005 Maxima of entries of Haar distributed matrices *Probab. Theory Relat. Fields* **131** 121–44
- [11] Lee P A and Stone A D 1985 Universal conductance fluctuations in metals *Phys. Rev. Lett.* **55** 1622–5
- [12] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol 2 (New York: McGraw-Hill)
- [13] Muirhead R J 1982 *Aspects of Multivariable Statistical Theory* (New York: Wiley)
- [14] Pichard J-L, Jalabert R A and Beenakker C W J 1994 Universal signatures of chaos in ballistic transport *Europhys. Lett.* **27** 255–60
- [15] Simon S H and Moustakas A L 2005 Crossover from conserving to lossy transport in circular random matrix ensembles *Preprint cond-mat/0511477*
- [16] Stone A D, Mello P A, Muttalib K A and Pichard J-L 1991 Random matrix theory and maximum entropy models for disordered conductors *Mesoscopic Phenomena in Solids* ed P A Lee, B L Altshuler and R A Webb (Amsterdam: North-Holland) pp 369–448
- [17] Verbaarschot J M, Weidenmüller H A and Zirnbauer M R 1985 Grassman integration in stochastic quantum physics: the case of compound nucleus scattering *Phys. Rep.* **129** 367–438
- [18] Washburn S and Webb R A 1986 Aharonov–Bohm effect in normal metal quantum coherence and transport *Adv. Phys.* **35** 375–422
- [19] Mahaux C and Weidenmüller H A 1969 *Shell-Model Approach to Nuclear Reactions* (New York: North-Holland)
- [20] Zyczkowski K and Sommers H-J 2000 Truncations of random unitary matrices *J. Phys. A: Math. Gen.* **33** 2045